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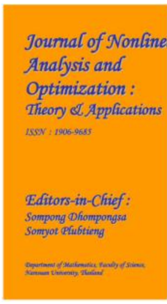

UNDERSTANDING DYNAMICS AND BIFURCATIONS ANALYSIS IN PREDATOR-PREY SYSTEMS

Biswajit Saha Assistant Professor Gobardanga Hindu College biswajitsaha.1987.bs@gmail.com
Orcid Id: 0000-0003-2690-7524

Abstract

Predator-prey interactions in ecological systems have been extensively studied due to their fundamental role in shaping ecosystem dynamics. In this study, we investigate the bifurcation analysis of predator-prey models using a variety of approaches and models proposed in the literature. Colon et al. (2015) presented an agent-based model for predator-prey interactions, analyzing bifurcations and their implications on ecosystem stability [1]. Hu.et al. (2017) explored stability and bifurcation analysis in a predator-prey system incorporating Michaelis Menten type predator harvesting [4]. Ryu. et al. (2018) focused on bifurcation analysis in a predator-prey system with functional responses increasing in both predator and prey densities [7]. Rana (2015) studied bifurcation and complex dynamics in a discrete-time predator-prey system [6], while Naik et al. (2023) performed bifurcation analysis of a discrete-time prey- predator model [5]. Additionally, Fussmann et al. (2000) contributed significant insights by analyzing the crossing of Hopf bifurcation in a live predator-prey system [2, 3].

Keywords Predator-prey dynamics, Lotka-Volterra model, Hopf bifurcation, System stability, Ecological sys tems.

Introduction

Ecological systems exhibit intricate patterns and behaviors driven by interactions between predator and prey populations. Understanding these dynamics lies at the heart of ecological research, presenting challenges that necessitate a multidisciplinary approach encompassing mathematics, biology, and ecology. The Lotka-Volterra model stands as a pivotal mathematical construct, providing a framework to describe the dynamics of predator-prey interactions. Developed independently by Alfred Lotka and Vito Volterra in the early 20th century, this model captures the essence of these interactions through coupled ordinary differential equations governing the populations of predators and prey [1, 4, 7, 6, 5, 2]. The Lotka-Volterra model's differential equations

$$dx/dt=x(b-x-y/(1+x))$$
$$dy/dt=y(x/(1+x)-ay)$$

form the cornerstone of our investigation. Here, x represents the prey population, y denotes the predator population, b signifies the prey's growth rate, and a denotes the predator's capturing efficiency. The model allows us to delve into the complex interplay between these populations, shedding light on the cyclic patterns, stability, and equilibrium states inherent in predator-prey systems.

This paper embarks on an exploratory journey into the mathematical intricacies of predator- prey dynamics[9], starting with a comprehensive overview of the Lotka-Volterra model's theoretical underpinnings. Through mathematical analysis and simulations, this study aims to unravel the system's behavior under varying parameters and conditions. The investigation particularly focuses on bifurcations, such as the critical Hopf bifurcation, which fundamentally alters the system's dynamics, leading to oscillatory behavior around equilibrium point. By amalgamating mathematical rigor with ecological relevance, this research endeavors to bridge the gap between theoretical models and empirical observations in ecological studies. The insights gleaned from this analysis offer valuable perspectives on ecosystem stability, resilience, and responses to environmental changes, elucidating the delicate dance between predator and prey populations in natural habitats.

Model Description

The predator-prey model is described by the following differential equations:

$$dx/dt = x(b - x - y/(1+x)) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$dy/dt = y(x/(1+x) - ay) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Here, the variables represent:

x, y: Population of prey species and Population of predator species.

b, a: Influences the prey's growth rate and interactions and Affects the predator's growth rate and dependency on prey.

These equations illustrate the dynamics of the predator-prey interaction, where the prey's growth and predation by the predator influence their populations over time. The critical value for the occurrence of Hopf bifurcation is given by:

$$a_c = (4(b-2))/(b^2 (b+2)) \quad \ldots\ldots\ldots(3)$$

This value helps identify conditions under which the system undergoes a transition from stable behavior to oscillatory behavior.

Understanding System Dynamics:

This step involves a comprehensive examination of how variations in the parameters a and b impact the dynamics of the predator-prey system.

For parameter a: Changes in a influence the predator's growth rate and its dependence on the prey population. Higher values of a typically lead to an increase in the predator's growth rate and a more significant impact on the prey population.

For parameter b: Alterations in b affect the prey's growth rate and interactions within the ecosystem. Higher values of b generally signify a faster growth rate for the prey, with implications for its interactions with the predator population.

By systematically varying a and b, this analysis phase aims to understand how these parameters individually and collectively shape the behavior and interplay between the predator and prey populations. It explores the effects of these variations on the stability, growth rates, and equilibrium points within the ecosystem.

Hopf Bifurcation Analysis:

This phase of analysis involves a focused investigation into the conditions that result in a transition from a stable system behavior to an oscillatory behavior within predator-prey dynamics. Hopf bifurcation occurs when certain parameter values reach critical thresholds, leading to qualitative changes in the system's behavior[8, 10]. It marks the point where the equilibrium state of the system changes its stability, causing oscillations or periodic solutions to emerge. In this analysis, specific parameter values, often denoted as critical values, are examined to identify when the system undergoes a shift from a steady state to oscillatory behavior. By studying the relationships between these parameters and observing their effects on the stability of the equilibrium points, this analysis aims to understand the occurrence and implications of the Hopf bifurcation in the predator-prey model.

Modeling Dynamics:

This phase involves the generation of visual representations, namely phase portraits and nullclines, to provide a comprehensive visualization of population trajectories and equilibrium points within the predator-prey model.

Phase Portraits: These graphical representations showcase the system's behavior by plot- ting the population of prey against the population of predators. Trajectories in the phase plane illustrate the dynamic interactions between the two populations over time. Different initial conditions or parameter values can lead to diverse trajectory patterns, aiding in the understanding of the system's behavior and its stability.

Nullclines: These curves represent the points where the rates of change of the prey and predator populations are zero. Specifically, the prey nullcline (dx/dt=0) and predator nullcline (dy/dt=0) are plotted on the phase plane. These curves help identify the equilibrium points, where the population sizes remain constant over time. Visualizing nullclines

along with trajectories offers insights into the system's dynamics, indicating how populations evolve and stabilize under various conditions.

By generating these visual representations, researchers can gain valuable insights into the behavior of the predator-prey model. Understanding the trajectories, equilibrium points, and the interplay between populations aids in comprehending the system's stability, predicting its behavior under different scenarios, and analyzing the effects of parameter variations on the ecosystem dynamics.

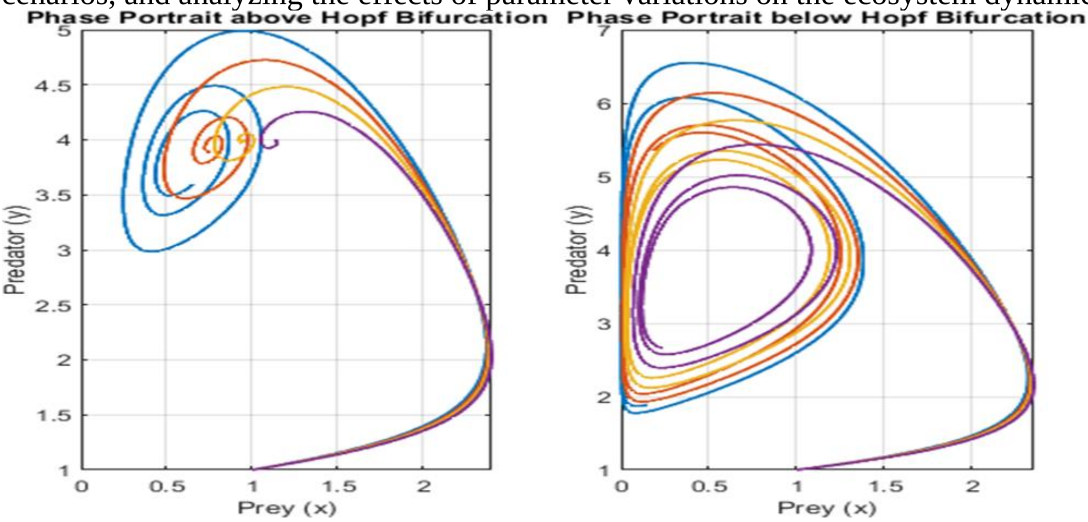

Figure 1: Phase Portrait above and below Bifurcation

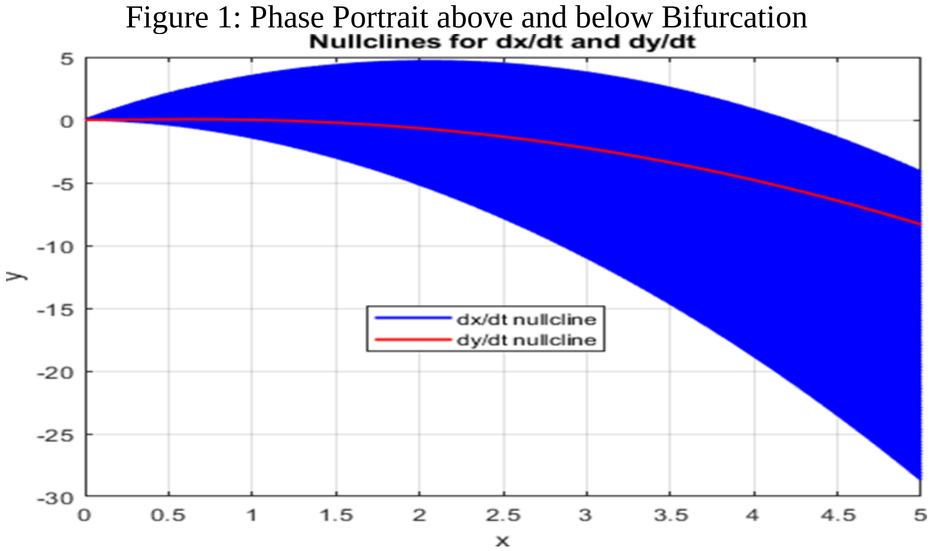

Figure 2: Nullclines for (dx/dt) and (dy/dt)

Step 3: Bifurcation Plot

A plot of the critical value ac against varying values of b is generated. The critical value ac is calculated as:

$a_c=(4(b-2))/(b^2 (b+2))$

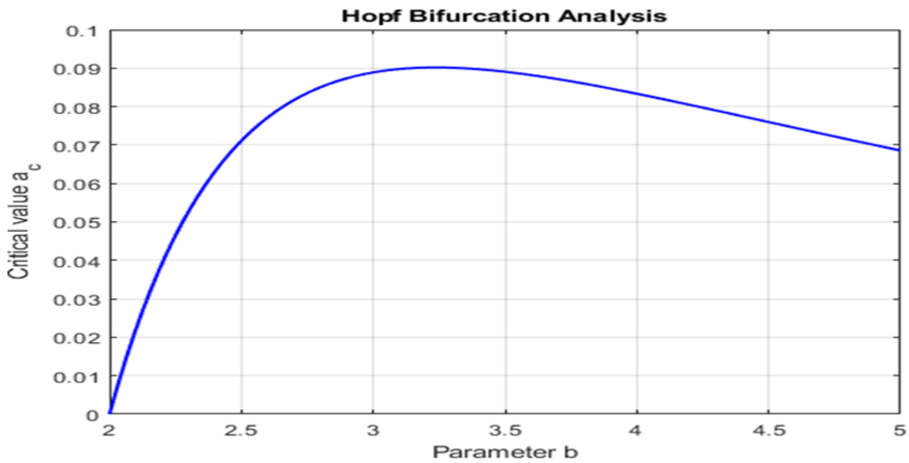

Figure 3: Bifurcation Analysis

Hopf bifurcation analysis is performed by plotting x∗ and ac against b.

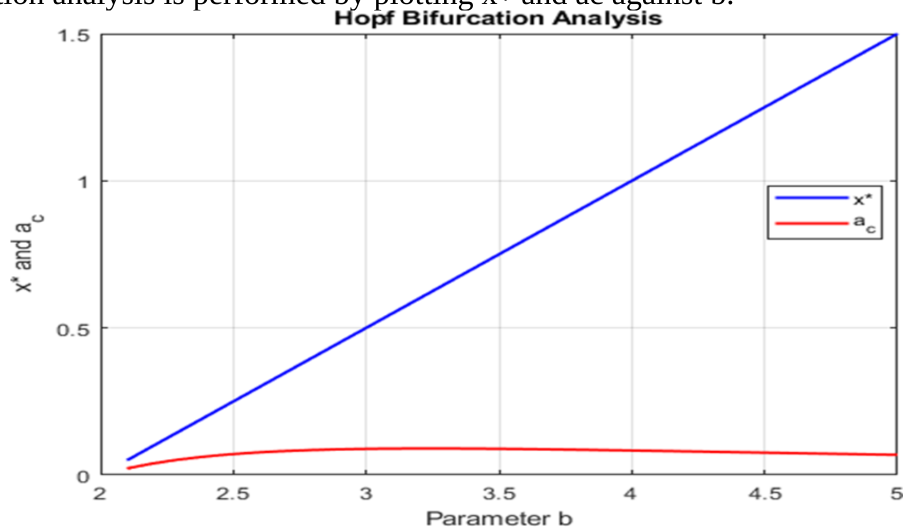

Figure 4: Bifurcation Analysis

Step 5: Phase Portraits

Phase portraits for different values of a are plotted, illustrating the dynamics of prey and predator populations over time.

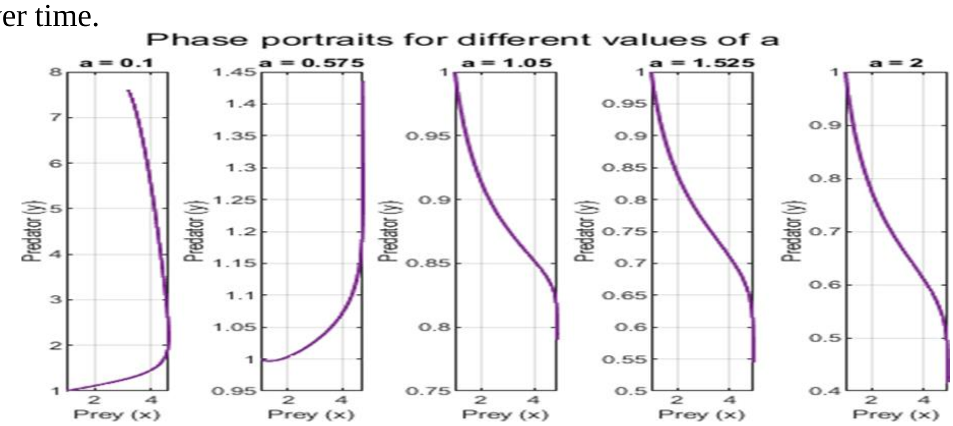

Figure 5: Phase Portraits

Step 6: Positive Fixed Point Existence Check

The code checks for positive fixed points (x∗, y∗) for various combinations of a and b within defined ranges. It determines whether a positive fixed point exists and displays its values if found.

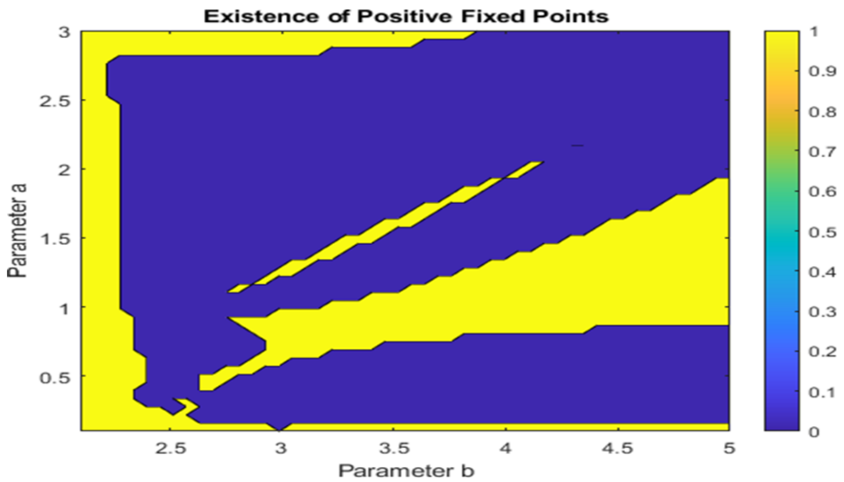

Figure 6: Positive Fixed Point Existence

Stability of the Function

The stability analysis of the predator-prey model is critical in understanding the behavior of the system and predicting its long-term dynamics. It involves determining the stability of the equilibrium points and assessing how the populations of prey and predators respond to perturbations around these points.

Equilibrium Point Analysis

The equilibrium points, denoted as $(x*, y*)$, represent the population sizes where the rates of change of the prey and predator populations are zero. The stability of these equilibrium points is assessed by examining the behavior of the system in response to small disturbances or deviations from these points.

For instance, an equilibrium point is considered:

Stable: If small perturbations lead the system back towards the equilibrium after disturbance, indicating resilience and stability in population dynamics.

Unstable: If small perturbations cause the system to move away from the equilibrium, it is suggested that deviations grow over time and the system does not return to the original state.

Semi-Stable or Saddle Node: When perturbations in certain directions lead to the system moving away from the equilibrium, while in other directions, the system returns towards the equilibrium.

Upon analysis using fsolve, the completion message indicates that the equilibrium point $(x* = 0.2, y* = 1.753 \times 10{-}8)$ has been found. However, it is reported to be unstable based on the system dynamics and behavior near this point. This instability suggests that small deviations from this equilibrium point may cause the system to move away rather than return to the original state.

Conclusion

In conclusion, this study delved into the intricate dynamics of predator-prey interactions using the Lotka-Volterra model as a foundational framework. Through mathematical analysis and simulation, we explored the system's behavior under varying parameters, shedding light on crucial aspects of ecological dynamics. The investigation into bifurcations, particularly the critical Hopf bifurcation, unveiled the pivotal points where the system transitions from stable to oscillatory be havior. This insight provides invaluable knowledge about the tipping points within predator-prey ecosystems and their implications for stability and resilience. Moreover, the modeling of system dynamics through phase portraits and nullclines offered a visual narrative of population trajectories and equilibrium points. These representations enhanced our understanding of the system's behavior, equilibrium states, and the impact of parameter variations on ecological stability. The stability analysis of equilibrium points highlighted the sensitivity of the system to perturbations, with the determination of the equilibrium point's instability offering crucial insights into the system's response to deviations.

This interdisciplinary approach, intertwining mathematical modeling with ecological principles, serves as a bridge between theoretical investigations and empirical observations. The findingsof this study contribute to our comprehension of ecological systems' responses to environmental changes, stability, and the delicate interplay between predator and prey populations in natural habitats. In summary, the research conducted offers a deeper understanding of predator-prey dynamics,emphasizing the importance of mathematical models in elucidating the underlying principles governing ecological systems.

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